# A soft introduction to TMF

Plan:
- The moduli stack of elliptic curves and modular forms
- A quick review of formal groups ~~and~~ the map $\mathcal{M}_{ell} \to \mathcal{M}_{FG}$
- Elliptic cohomology via elliptic formal groups, the pushleaf $\mathcal{M}_{ell\, \acute{e}t} \to \begin{Bmatrix} \text{cohomology} \\ \text{theories} \end{Bmatrix}$
- The Goerss – Hopkins – Miller – Lurie theorem

## The moduli stack of elliptic curves and modular forms

Elliptic curves over $\mathbb{C}$ are quotients $\mathbb{C}/\Lambda$ $\Lambda$ rank 2 lattice $\Lambda = \mathbb{Z} \oplus \tau\mathbb{Z}$, $\tau \in \mathcal{H}$ upper-half plane.

We want to classify these objects. One observes that

$$E_\tau = \mathbb{C}/\Lambda_\tau \simeq E_{\tau'} \quad \text{if } \tau \text{ and } \tau' \text{ are related by}$$

$$\tau' = M \cdot \tau \qquad M \in SL(2;\mathbb{Z}) \text{ acts on } z \in \mathbb{C} \text{ by}$$

$$M \cdot z = \frac{az+b}{cz+d} \qquad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

This means that the object classifying complex elliptic curves is the "orbifold"

$$\mathcal{M}_{ell,\mathbb{C}} = \mathcal{H}/_{SL(2;\mathbb{Z})}$$

or "quotient stack" in algebraic geometry.

One might desire, as we work in AG, to extend this object from $\mathbb{C}$ to $\mathbb{Z}$.

**Def**: let $S$ be a scheme. An elliptic curve over $S$ is a proper smooth scheme $E \xrightarrow{p} S$ over $S$ with a map $1: S \to E$ which is a section of $p$ and such that for every algebraically closed field $K$, the fiber

$$E_K = E \times_S \operatorname{Spec} K \text{ for any map } \operatorname{Spec} K \to S$$

is an algebraic curve of genus 1 ( $p$ is smooth so arithmetic genus = genus = $\dim_K$ ~~sec~~ vector space of sections of $\Omega^1_{E_K}$ )

Moduli stack of elliptic curves is the functor

$$\begin{aligned} \mathcal{M}_{ell} : \mathrm{Aff}^{op} &\longrightarrow \mathcal{S} \\ S &\longmapsto \left\{ \begin{array}{c} \text{elliptic curves} / S \\ + \text{isomorphisms} \end{array} \right\} \end{aligned}$$

and $\mathcal{M}_{ell\,\mathbb{C}} = \mathcal{M}_{ell} \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{C}$

**Rmk**: elliptic curves over $\mathbb{C}$ are abelian groups, as they are $\mathbb{C}/\Lambda$. This extends to $E \xrightarrow{p} S$ which acquire the structure of groups in $\mathrm{Sch}_{/S}$. In particular we can consider $\mathrm{Lie}(E)$; $\mathrm{Lie}(E_{\mathbb{C}})^{\vee} = \Omega^1_{E,\cancel{1}} \simeq \mathbb{C}$.

This extends to a line bundle $\omega$ over $\mathcal{M}_{ell}$, $\omega|_S = p_* \Omega^1_{E/S}$

**Def**: the ~~vector space~~ abelian group of modular forms ~~of~~ is the group

$$MF = \bigoplus_{k \in \mathbb{Z}} H^0(\mathcal{M}_{ell}, \mathcal{O} \otimes_{\mathcal{O}} \omega^k)$$

these are $\neq 0$ only for $k \geq 0$ ).

# The stack of formal groups $\mathcal{M}_{FG}$, the map $\mathcal{M}_{ell} \to \mathcal{M}_{FG}$, and elliptic cohomology

Quick reminder of Alessandro's talk: there is a dictionary

$$\text{complex oriented cohomology theories} \longleftrightarrow \text{1 dimensional (smooth) commutative associative formal group laws}$$

modulated by $\mathbb{CP}^\infty$.

<u>Def</u>: a formal group over a scheme $S$ is a commutative group object in formal schemes$/S$ equipped with $\times_S$ monoidal structure.

Here formal schemes can be though of as ind-schemes.

A (smooth) formal group is equivalently a formal group law up to isomorphism.

Moduli stack of formal groups laws is

$$\mathcal{M}_{FGL} = \operatorname{Spec} L \quad , L \text{ Lazard ring:}$$

$$\{\text{fgl's on } R\} = \operatorname{Hom}_{\text{Rings}}(L, R) \qquad L \simeq \mathbb{Z}[x_1, x_2, \ldots]$$

More geometrically, $L \simeq MP^*_*(pt)$ periodic complex cobordism.

An automorphism of a fgl $F(x,y)$ is a formal power series

$$g(t) = ut + \sum_{i \geq 2} a_i t^i \quad \text{with } u \text{ invertible}$$

acting as $(g \cdot F)(x,y) = g^{-1}\left(F(g^{-1}(x), g^{-1}(y))\right)$

Then $\mathcal{M}_{FG} \simeq \mathcal{M}_{FGL}/\text{automorphisms}$

The group of automorphisms is $H : \mathrm{Aff} \longrightarrow \mathcal{S}$

$R \longmapsto \left\{ \text{power series } ux + \sum_{i \geq 2} a_i x^i,\ u \in R^\times,\ a_i \in R \right\}$

$\left( \text{also is } MP_0(MP),\ \underset{M_{FG}}{\overset{\operatorname{Spec} MP_0(MP)}{\operatorname{Spec} L \times \operatorname{Spec} L}} \right)$

~~$\simeq \mathrm{Aut}$~~

$E \to S$ be an elliptic curve over $S$. We construct a formal group by taking formal completion at the identity section $1 : S \to E$, $\hat{E}$. This is a formal group as $E$ is a group.

The assignment $E \longmapsto \hat{E}$ "lifts" to a map

$$\mathcal{M}_{ell} \longrightarrow \mathcal{M}_{FG} \qquad \left( \begin{array}{c}\text{Hopkins - Miller} \\ \text{this map is flat}\end{array} \right)$$

Reminder on Landweber exactness: when does a formal group give rise to a cohomology theory?

$R$ ring. Choose ~~fgl~~ $F$ with coefficients in $R$:

$$F : L \simeq MP_0(pt) \longrightarrow R.$$

$X$ topological space, does $X \longmapsto MP_*(X) \otimes_{MP_0(pt)} R$ give a cohomology theory?

Turns out we do not have Mayer-Vietoris in general.

$$\underbrace{x +_{MU} x +_{MU} \cdots +_{MU} x}_{p \text{ prime}} = \sum_{k=0}^{\infty} a_k x^{k+1} \qquad v_i := a_{p^i - 1}$$

$$I_{p,n} = (p, v_1, \ldots, v_{n-1})$$

$F : L \to R$, if $v_n : R/I_{p,n} \to R/I_{p,n}$ injective $\forall p, n$, OK.

In more modern terms: $\mathcal{M}_{FG} = \mathcal{M}_{(MP_0, MP_0(MP))}$.

$\mathrm{Qcoh}(\mathcal{M}_{FG}) = \mathrm{CoMod}_{(MP_0, MP_0(MP))}$. For $F: \mathrm{Spec}\, R \to \mathcal{M}_{FG}$,

$\underbrace{F^* : \mathrm{Qcoh}(\mathcal{M}_{FG}) \longrightarrow \mathrm{Qcoh}(\mathrm{Spec}\, R) \text{ is exact}}$ iff $F$ flat.

is Landweber exactness; for $X$, $MP_*(X)$ is a $(MP_0, MP_0(MP))$-comodule i.e. is in $\mathrm{Qcoh}(\mathcal{M}_{FG})$.

Landweber exactness, modern: $F$ is LE if $\mathrm{Spec}\, R \xrightarrow{F} \mathcal{M}_{FG}$ is flat.

Remark: $\mathcal{M}_{ell} \to \mathcal{M}_{FG}$ is flat: if $E$ elliptic curve over $R$ has

$E: \mathrm{Spec}\, R \longrightarrow \mathcal{M}_{ell}$ flat, then

$$\hat{E}: \mathrm{Spec}\, R \longrightarrow \mathcal{M}_{FG}$$

is LE, i.e. yields a complex-oriented (periodic) cohomology theory via

$$E(X) = MP_*(X) \otimes_{MP_0(pt)} R$$

where $\hat{E}: MP_0(pt) \to R$.

This is the <u>elliptic cohomology theory</u> associated to $E$.

This gives us an assignment

$$\left\{ \mathrm{Spec}\, R \xrightarrow[\text{flat}]{E} \mathcal{M}_{ell} \right\}^{op} \longrightarrow \left\{ \begin{array}{c} \text{complex-oriented} \\ \text{cohomology theories} \end{array} \right\}$$

$$\begin{array}{ccc} \| & & \\ |\mathcal{M}_{ell}|_{flat} & \begin{array}{c} E \\ \downarrow \\ \mathrm{Spec}\, R \end{array} \longmapsto & \text{coh. thy associated to } \hat{E} \end{array}$$

"small" flat site of $\mathcal{M}_{ell}$

<u>Recap</u>: via Landweber exactness we obtain a presheaf, for $X$ CW-complex

$$\begin{array}{ccc}
\mathrm{Aff}^{\text{flat}}/_{\mathcal{M}_{ell}} & \longrightarrow & \text{cohomology theories} \\
\rotatebox{90}{$\in$} & & \\
\mathrm{Spec}\, R \xrightarrow{\phi} \mathcal{M}_{ell} & \longmapsto & MP_*(X) \otimes_{MP_0(pt)} R \\
\phi \text{ flat} & &
\end{array}$$

where $L = MP_0(pt) \longrightarrow R$ is the module structure given by the fgl $\hat{E}$, $E$ classified by $\phi$

<u>Thm</u> (Goerss–Hopkins–Miller): there is a lift

$$\begin{tikzcd}
 & \text{commutative ring spectra} \arrow[d] \\
\mathrm{Aff}^{\acute{e}t}/_{\mathcal{M}_{ell}} \arrow[ur, dashed, "\mathcal{O}^{top}"] \arrow[r] & \text{cohomology theories}
\end{tikzcd}$$

$$\mathrm{Aff}^{\acute{e}t}/_{\mathcal{M}_{ell}} = \mathcal{M}_{ell}^{\acute{e}t}$$

Moreover, $\mathcal{O}^{top}$ is a <u>sheaf</u>: $U = \mathrm{Spec}\, A$, $U = \bigcup U_\alpha$

$\Rightarrow \mathcal{O}^{top}(U) = \lim \mathcal{O}^{top}(U_\alpha)$

In particular, we now have $\mathcal{O}^{top}(\mathcal{M}_{ell}) \in$ commutative ring spectra

(the sheaf property tells us how to extend to the big étale site of $\mathcal{M}_{ell}$)

**Def**: $TMF := \mathcal{O}^{top}(\mathcal{M}_{ell})$.

**Rmk**: the proof of GHM theorem is based on very hard obstruction theory totally unknown to me, developed in about 500 pages of math.

Jacob Lurie's role in this story is the following: in his PhD thesis he introduces DAG (now known as SAG) as a framework in which $(\mathcal{M}_{ell}, \mathcal{O}^{top})$ should naturally live.

**Some facts**: • $\underbrace{\pi_{2n}(\mathcal{O}^{top})}_{\text{homotopy sheaves}} = \omega^n$

• $\pi_*(\mathbb{S}) \xrightarrow{②} \pi_*(tmf) \xrightarrow{①} MF_*$

here $tmf = \tau_{\geq 0} Tmf$, $Tmf = (\overline{\mathcal{M}_{ell}}, \mathcal{O}^{top})$'s global sections.

② Has information about torsion ($\mathbb{S} \to tmf$)

① is an isomorphism after inverting 2 and 3, and

$$\pi_{2n}(tmf) \longrightarrow MF_n \qquad \forall n$$

$$\pi_{2n+1}(tmf) \longrightarrow 0$$

This map comes from the "elliptic spectral sequence" ie descent spectral sequence

$$H^s(\overline{\mathcal{M}_{ell}}, \omega^{2n}) \Rightarrow \pi_{2n-s}(Tmf)$$

Rmk : you have this spectral sequence in general: for $F$ a sheaf of spectra on a stack $X$, (8)

$$H^s(X; \pi_n(F)) \Rightarrow \pi_{n-s}(\Gamma(X;F)).$$

tmf is not periodic, but TMF is. Indeed

$$\pi_*(TMF) = \pi_*(tmf)[(\Delta^{24})^{-1}], \quad |\Delta| = 24$$

$\Rightarrow$ $\cdot \Delta^{24}$ is an isomorphism $\pi_*(TMF) \xrightarrow{\sim} \pi_{*+576}(TMF)$

$\Delta$ is a modular form, $\Delta(z) = (2\pi)^{12}\, \eta(z)^{24}$

"modular discriminant" $\quad \eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n)$

$\Delta$ is a modular form of weight 12 (i.e $\Delta$ is a section of $\omega^{12}/\mathbb{Z}$, i.e. of $\pi_{24}(\mathcal{O}^{top})$).

Explicitly, a modular form of weight $k$ has $f\left(\frac{az+b}{cz+b}\right) = f(z)\cdot(cz+d)^k$

Aside : Lurie's approach. Idea : have a spectral moduli stack $(\mathcal{M}_{ell}^{or}, \mathcal{O}^{top})$ of oriented elliptic curves which truncates to the classical one.

Def: Let $R \in CAlg$ be an $E_\infty$-ring. A variety over $R$ is a nonconnective spectral ~~scheme~~ DM stack $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ flat over $\mathrm{Sp\acute{e}t}\, R$ s.t.

$(\mathcal{X}, \tau_{\geq 0}\mathcal{O}_{\mathcal{X}}) \longrightarrow \mathrm{Sp\acute{e}t}\, \tau_{\geq 0} R$ is proper, locally almost finite presentation, geometrically reduced and geometrically connected.

Abelian varieties are commutative monoids in $Var(R)$,

$AVar(R) = CMon(Var(R))$

Strict abelian varieties are abelian group objects in $\mathrm{Var}(R)$, i.e. finite products preserving functors from the category of lattices $\mathrm{Lat}^{op} = \{\text{finite rank free } \mathbb{Z}\text{-modules}\}$ to $\mathrm{Var}(R)$; $\mathrm{AVar}^{S}(R) = \mathrm{Ab}(\mathrm{Var}(R))$

Here $\mathrm{Lat}$ is the Lawvere theory of abelian groups:

$$\mathrm{Fun}^{\pi}(\mathrm{Lat}^{op}, \mathrm{Set}) \simeq \mathrm{Ab}$$

Replacing Set with $\mathcal{S}$ we get $\mathrm{Ab}(\mathcal{S}) = \mathcal{P}_{\Sigma}(\mathrm{Lat})$, the "sifted completion" of Lat. By observing that $\mathrm{Lat} \hookrightarrow \mathrm{Mod}_{\mathbb{Z}}^{cn}$ (connective $\mathbb{Z}$-modules), we get $\mathrm{Ab}(\mathcal{S}) \to \mathrm{Mod}_{\mathbb{Z}}^{cn}$ which is an equivalence. (via left Kan extension).

<u>Rmk</u>: we have a functor of points construction

$$\mathrm{Var}(R) \longrightarrow \mathrm{Fun}(\mathrm{CAlg}_R, \mathcal{S})$$

Then

- $\mathrm{AVar}(R) \overset{\circledast}{=} \mathrm{CMon}^{gp}(\mathrm{Var}(R)) \hookrightarrow \mathrm{CMon}^{gp}(\mathrm{Fun}(\mathrm{CAlg}_R, \mathcal{S}))$
  $= \mathrm{Fun}(\mathrm{CAlg}_R, \mathrm{CMon}^{gp}(\mathcal{S})) = \mathrm{Fun}(\mathrm{CAlg}_R, \mathrm{Sp}^{cn})$

$\circledast$ all abelian varieties are grouplike

- $\mathrm{Ab}(\mathrm{AVar}(R)) = \mathrm{Ab}(\mathrm{CMon}(\mathrm{Var}(R))) = \mathrm{CMon}(\mathrm{Ab}(\mathrm{Var}(R))) =$
  $\simeq \mathrm{Ab}(\mathrm{Var}(R)) = \mathrm{AVar}^{S}(R)$
- $\mathrm{AVar}^{S}(R) \hookrightarrow \mathrm{Ab}(\mathrm{Fun}(\mathrm{CAlg}_R, \mathcal{S})) \simeq \mathrm{Fun}(\mathrm{CAlg}_R, \mathrm{Ab}(\mathcal{S}))$
  $\simeq \mathrm{Fun}(\mathrm{CAlg}_R, \mathrm{Mod}_{\mathbb{Z}}^{cn})$

<u>Rmk</u>: if $R$ is an ordinary commutative ring, the forgetful map $\mathrm{AVar}^{S}(R) \longrightarrow \mathrm{AVar}(R)$ is an equivalence as $\mathrm{Var}(R)$ becomes a 1-category.

We always have a forgetful functor $\mathrm{Ab}(\mathcal{C}) \to \mathrm{CMon}(\mathcal{C})$, induced by $\mathrm{Fin}_* \to \mathrm{Lat}$. For a finite pointed set $I_*$, the set

$$I_* \longmapsto \mathrm{Hom}_*(I_*, \mathbb{Z})$$

of pointed maps $\mathrm{Hom}_*(I_*, \mathbb{Z})$ is a lattice.

For a variety $X$ over $R$ we have a notion of $\dim X$, which is the Krull dimension of $\mathrm{Sp\acute{e}t}\, \kappa \times_{\mathrm{Sp\acute{e}t}\, \pi_0 R} (\mathcal{X}, \pi_0 \mathcal{O}_{\mathcal{X}})$ for every field $\kappa$ and map $\pi_0 R \to \kappa$ (just as in the ordinary setting).

An (strict) elliptic curve $/R$ is an (strict) abelian variety over $R$ of dim. 1.

There exists a moduli stack of strict elliptic curves $\mathcal{M}^s_{\mathrm{Ell}}$ whose functor of points is $R \longmapsto \mathrm{Ell}^s(R)^{\simeq}$, i.e. the groupoid of strict elliptic curves over $R$.

We now upgrade this to a moduli stack of <u>oriented elliptic curves</u>.

<u>Def</u> (ELL II, 4.3.1): $R$ $E_\infty$-ring, $X : \mathrm{CAlg}^{cn}_{\tau_{\geq 0} R} \to \mathcal{S}_*$ a pointed formal hyperplane over $R$. A preorientation of $X$ is a pointed map

$$e : S^2 \longrightarrow X(\tau_{\geq 0} R)$$

- $\mathrm{Pre}(X) = \Omega^2 X(\tau_{\geq 0} R)$.

<u>Think</u>: a formal hyperplane is a generalization to $E_\infty$-rings of $\mathrm{Spf}\, R[[t_1, \dots, t_n]]$. See appendix.

We always consider (in classical settings) smooth connected commutative formal groups, i.e. formal groups whose underlying formal scheme is isomorphic to $\mathrm{Spf}\, R[[t_1, \dots, t_n]]$. Here we require the underlying formal scheme to be a formal hyperplane.

In this context, to a strict abelian variety we associate a formal group via

$$X \longmapsto \hat{X}, \text{ formal completion at the identity}$$

$$\hat{X} : \mathrm{CAlg}^{cn}_{\tau_{\geq 0} R} \longrightarrow \mathrm{Mod}_{\mathbb{Z}}^{cn}$$

$$A \longmapsto \mathrm{fib}\left( X(A) \longrightarrow X(A^{red}) \right)$$

<u>Def</u> (ELL II, 7.2.1): let $R$ be an $E_\infty$-ring and $X$ an strict elliptic curve over $R$. A preorientation of $X$ is a pointed map

$$e : S^2 \longrightarrow \Omega^\infty X(\tau_{\geq 0} R)$$

$$\mathrm{Pre}(X) = \Omega^2 \Omega^\infty X(\tau_{\geq 0})$$

<u>Rmk</u>: the canonical map

$$\hat{X}(\tau_{\geq 0} R) \longrightarrow X(\tau_{\geq 0} R)$$

induces an equivalence $\mathrm{Pre}(X) \simeq \mathrm{Pre}(\hat{X})$.

Here, $\mathrm{Pre}(\hat{X})$ are maps $e : S^2 \to \Omega^\infty \hat{X}(\tau_{\geq 0} R)$ which are preorientations of the underlying formal hyperplane

$$\mathrm{CAlg}^{cn}_{\tau_{\geq 0} R} \xrightarrow{\hat{X}} \mathrm{Mod}_{\mathbb{Z}}^{cn} \xrightarrow{\Omega^\infty} \mathcal{S}$$

There is a spectral Deligne-Mumford stack of preoriented elliptic curves $\mathcal{M}_{ell}^{pre}$ which embeds in $\mathcal{M}_{ell}^{s}$ as a closed. In particular, they have the same underlying classical DM stack $\mathcal{M}_{ell}$.

<u>Def</u>: a preorientation $e: S^2 \to X(\tau_{\geq 0} R)$ of a strict elliptic curve $X$ is an <u>orientation</u> if the induced preorientation of $\hat{X}$ is an orientation, i.e. the associated Bott map

$$\beta_e : \omega_{X,\eta} \longrightarrow \Sigma^{-2} R$$

is an equivalence.

The Bott map for a 1-dimensional formal hyperplane $Y$ with basepoint $\eta \in Y(\tau_{\geq 0} R)$ is constucted ~~as follows~~. in the appendix.

If $\mathrm{Ell}^{or}(R)^{\simeq}$ denotes the full subcategory of the category of preoriented elliptic curves $\mathrm{Ell}^{pre}(R)^{\simeq}$, the functor

$$\begin{aligned} \mathcal{M}_{ell}^{or} : \mathrm{CAlg}_{*} &\longrightarrow \mathcal{S} \\ R &\longmapsto \mathrm{Ell}^{or}(R) \end{aligned}$$

is a spectral DM stack. The natural map

$$\mathcal{M}_{ell}^{or} \xrightarrow[\text{affine}]{} \mathcal{M}_{ell}^{pre} \xrightarrow[\text{closed,}]{} \mathcal{M}_{ell}^{s}$$

(affine, locally finite presentation)

In particular,

$$\begin{tikzcd} \mathrm{Spec}\, R[\beta_e^{-1}] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{M}_{ell}^{or} \arrow[d] \\ \mathrm{Spec}(R) \arrow[r] & \mathcal{M}_{ell}^{pre} \end{tikzcd}$$

The map $\phi: \mathcal{M}_{ell}^{or} \to \mathcal{M}_{ell}^{s}$ gives a map $\mathcal{O}_{\mathcal{M}_{ell}^{s}} \longrightarrow \phi_* \mathcal{O}_{\mathcal{M}_{ell}^{or}}$, which induces an isomorphism $\pi_0 \mathcal{O}_{\mathcal{M}_{ell}^{s}} \simeq \pi_0 \mathcal{O}_{\mathcal{M}_{ell}^{or}}$. In particular, $\mathcal{M}_{ell}^{or}$ has $\mathcal{M}_{ell}$ as underlying $\infty$-topos. Moreover,

$$\pi_i(\mathcal{O}_{M_{ell}^{or}}) = \begin{cases} \omega^{\otimes k} & k=2j \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

Then $M_{ell}^{or}$ "gives" the Goerss – Hopkins – Miller theorem, i.e. it gives a canonical structure of spectral DM stack to $M_{ell}$. GHM also prove uniqueness up to homotopy while Lurie's construction gives a canonical lift.

<u>Def</u> : the periodic spectrum of topological modular forms is

$$TMF = \Gamma\left(M_{ell}^{or}, \mathcal{O}_{M_{ell}^{or}}\right).$$

<u>Appendix</u>

• <u>Formal hyperplanes</u>

These are generalizations of $\mathrm{Spf}\, R[[t_1, \dots, t_n]]$.

Let $R$ be a connective $E_\infty$-ring and $C$ a flat commutative coalgebra over $R$. For a map of connective $E_\infty$-rings $R \to A$ we have that

$$\begin{aligned} \mathrm{Mod}_R &\longrightarrow \mathrm{Mod}_A \\ M &\longmapsto A \otimes_R M \end{aligned}$$

is symmetric monoidal, so a coalgebra $C$ over $R$ is mapped to a coalgebra $C \otimes_R A$ over $A$. Define

$$\begin{aligned} c\mathrm{Spec}(C) : \mathrm{CAlg}_R^{cn} &\longrightarrow \mathcal{S} \\ A &\longmapsto \mathrm{Map}_{c\mathrm{CAlg}_A}(A, A \otimes_R C) \end{aligned}$$

This extends to a functor

$$\mathrm{cSpec} : \mathrm{cCAlg}^{b}_{R} \longrightarrow \mathrm{Fun}(\mathrm{CAlg}^{cn}_{R}, \mathcal{S})$$

for $R$ nonconnective, replace here by $\tau_{\geq 0}R$

Rmk \ Prop : $\mathrm{cSpec}(C) \simeq \mathrm{Spf}(C^{\vee})$ as soon as $C$ is smooth over $R$.

Prop (ELL2, 1.5.9) : let $R$ be an $E_\infty$-ring. We have a fully faithful embedding

$$\mathrm{cSpec} : \mathrm{cCAlg}^{sm}_{R} \hookrightarrow \mathrm{Fun}(\mathrm{CAlg}^{cn}_{\tau_{\geq 0}R}, \mathcal{S})$$

Def (ELL2, 1.5.9) : let $R \in \mathrm{CAlg}^{cn}$. A functor

$$X : \mathrm{CAlg}^{cn}_{R} \longrightarrow \mathcal{S}$$

is a formal hyperplane over $R$ if it belongs to the essential image of cSpec:

$$\begin{tikzcd} \mathrm{cSpec} : \mathrm{CAlg}^{cn}_{R} \arrow[r, "\sim"] \arrow[dr] & \mathrm{Hyp}(R) \arrow[d, hook] \\ & \mathrm{Fun}(\mathrm{CAlg}^{cn}_{R}, \mathcal{S}) \end{tikzcd}$$

• The Bott map

Def (ELL 2, 4.2.1) : for $R \in \mathrm{CAlg}$, $X$ a 1 dimensional formal hyperplane over $R$ with a basepoint $\eta \in X(\tau_{\geq 0}R)$ classified by a map $\epsilon : \mathcal{O}_X \to R$, the dualizing line is

$$\omega_{X,\eta} := \underbrace{\mathrm{fib}\left(\epsilon : \mathcal{O}_X \to R\right)}_{=: \mathcal{O}_X(-\eta)} \otimes_{\mathcal{O}_X} R$$

Rmk: there is a map $\rho : \omega_{X,\eta} \to L_{X/\mathrm{Spec}R, \eta}$ which is an equivalence if

$R$ is a $\mathbb{Q}$-algebra, and an isomophism on $\pi_0$ for connective $R$.

Construction (4.2.9): $R \in \mathrm{CAlg}$, $X$ 1 dim formal hyperplane, $\eta \in X(\tau_{\geq 0} R)$.
If $R$ is connective and $A$ is a connective $R$-algebra:

$$\Omega X(A) \simeq \mathrm{Map}_{\mathrm{CAlg}_R}(R \otimes_{\mathcal{O}_X} R, A) \to \mathrm{Map}_{\mathrm{Mod}_R}(R \otimes_{\mathcal{O}_X} R, A)$$

$$\to \mathrm{Map}_{\mathrm{Mod}_R}(\Sigma \omega_{X,\eta}, A) \xrightarrow{\sim} \Omega \mathrm{Map}_{\mathrm{Mod}_R}(\omega_{X,\eta}, A)$$

Here $\eta \in X(\tau_{\geq 0} R)$ gives a map $\epsilon: \mathcal{O}_X \to R$ which makes $R$ into an $\mathcal{O}_X$-algebra. Then $\Sigma \omega_{X,\eta}$ is the fiber of the multiplication $m: R \otimes_{\mathcal{O}_X} R \to R$. The map

$$\Omega X(A) \longrightarrow \Omega \mathrm{Map}_{\mathrm{Mod}_R}(\omega_{X,\eta}, A)$$

is called <u>linearization map</u>. By taking $\Omega$ we get ~~Bott map~~ (also $A = \tau_{\geq 0} R$)

$$\Omega^2 X(\underset{\tau_{\geq 0} R}{\overset{\shortparallel}{A}}) \cong \mathrm{Pre}(X) \longrightarrow \Omega^2 \mathrm{Map}_{\mathrm{Mod}_R}(\omega_{X,\eta}, \underbrace{\tau_{\geq 0} R}_{\text{can be replaced by } R})$$

$$\simeq \mathrm{Map}_{\mathrm{Mod}_R}(\omega_{X,\eta}, \Sigma^{-2} \tau_{\geq 0} R)$$

$$e \longmapsto \beta_e \text{ the Bott Map of } e$$

<u>Some comments on this story</u>:

Classically, to an even periodic complex oriented cohomology theory we can attach a formal group, the classical Quillen formal group

$$\mathrm{Spf}(A^0(\mathbb{C}P^\infty))$$

Complex-orientability of $A$ implies that it is smooth and connected, and a choice of orientation gives an isomorphism

$$\mathrm{Spf}\, A^0(\mathbb{C}P^\infty) \xrightarrow{\sim} \mathrm{Spf}\, A^0(pt)[[t]]$$

For an $E_\infty$-ring $A$, we can enhance this classical formal group to a "derived" one

$$\mathrm{Spf}\, \underbrace{C^*(\mathbb{CP}^\infty, A)}_{\substack{\text{spectrum of maps}\\ \mathbb{CP}^\infty \to A}} = \hat{\mathbb{G}}_A^Q$$

, its $\pi_0$ is $A^0(\mathbb{CP}^\infty)$

The formal group structure comes (as in the classical case) from observing that $\mathbb{CP}^\infty \in \mathrm{Ab}(\mathcal{S}_*)$ (by the map classifying tensor product of line bundles). As abelian group in $\mathcal{S}$, $\mathbb{CP}^\infty$ is freely generated by $S^2$:

$$\mathrm{Map}_{\mathrm{Ab}(\mathcal{S})}(\mathbb{CP}^\infty, M) \underset{\substack{\uparrow \\ S^2 \simeq \mathbb{CP}^1 \hookrightarrow \mathbb{CP}^\infty}}{\simeq} \mathrm{Map}_{\mathcal{S}_*}(S^2, M)$$

This means that also $\hat{\mathbb{G}}_A^Q$ is "freely generated by $S^2$" as formal group: ~~for~~ any map

$$f : \hat{\mathbb{G}}_A^Q \longrightarrow \hat{\mathbb{G}}$$

to a formal group $\hat{\mathbb{G}}$ is equivalent to a map

$$e_f : S^2 \longrightarrow \Omega^2 \hat{\mathbb{G}}(\tau_{\geq 0} R)$$

Such a map of formal groups will be an isomorphism if "its derivative at the identity" is an isomorphism. This is replaced by

$$f^* : \omega_{\hat{\mathbb{G}}} \longrightarrow \omega_{\hat{\mathbb{G}}_A^Q}$$

being an equivalence, as $\omega_{\hat{\mathbb{G}}}$ plays the role of the cotangent complex in this context. Finally, one has

$$\omega_{\hat{\mathbb{G}}_A^Q} \simeq \Sigma^{-2} A$$

obtaining the notion of orientation.