SPECTRA, COMPLEX ORIENTED COHOMOLOGY THEORY, AND FORMAL GROUP LAWS

These are notes for a talk I gave at the Max-Planck Institute for Mathematics in Bonn, for the "Physics Math" seminar. The abstract to the talk can be found here. If you find mistakes in the notes (hopefully there aren't too many), please contact me at alessandro.nanto3@gmail.com.

1. Background and Notations

We begin by fixing some notation.

1.1. **Topological spaces.**

Denote by $\mathcal{T}op$ a *convenient* category of topological spaces, for example weakly Hausdorff, compactly generated spaces, also called *k-spaces* (see [10]). A convenient category of topological spaces is characterized, amongst other things, by the fact that $-\times X : \mathcal{T}op \to \mathcal{T}op$ has a right adjoint $\widetilde{\mathrm{Hom}}(X,-) : \mathcal{T}op \to \mathcal{T}op$ given by mapping Y to the space of continuous maps $X \to Y$ with compact-open topology. Every unbased space X can be turned based by adding a disjoint single point, denote by X_+ the based space $X \sqcup \{*\}$. The functor $-_+ : \mathcal{T}op \to \mathcal{T}op_*$ is left adjoint to the functor $\mathcal{T}op_* \to \mathcal{T}op$ forgetting the basepoint.

For pointed spaces, we denote products by $\wedge$. Explicitly, given pointed spaces (X,x) and (Y,y), then $X \wedge Y$ is the product $X \times Y$ with the subspace $X \times \{y\} \cup \{x\} \times Y$ collapsed to a point (which is then taken as the basepoint). We also denote coproducts by $\vee$. Explicitly $X \vee Y$ is the pushout of the diagram

$$\begin{array}{ccc} * & \longrightarrow & Y \\ \big\downarrow & & \\ X & & \end{array}$$

(where $*$ is the one-point space). Equivalently, it is $X \sqcup Y$ with $\{x,y\}$ collapsed to a point.

Given a pointed space (X,x), just like in unbased pointed spaces, the functor $-\wedge X$ has a right adjoint $\widetilde{\mathrm{Hom}}_*(X,-)$, which maps a pointed space (Y,y) to the subspace of $\widetilde{\mathrm{Hom}}(X,Y)$ of basepoint preserving continuous functions (with the subspace topology).

Remark 1.1. Given points $z \in X, w \in Y$, we'll write $z \wedge w$ to mean the point in $X \wedge Y$ represented by the pair $(z,y) \in X \times Y$. In particular, if either z or w is the basepoint of the respective spaces, $z \wedge w$ is always the basepoint of $X \wedge Y$.

Remark 1.2. For the *n-sphere* S^n we always take the one-point compactification of $\mathbb{R}^n$, the point at ∞ acting as the base point. This is homeomorphic to the usual definition of S^n, but has the advantage that linear group actions on $\mathbb{R}^n$ automatically lift to based actions on S^n. Moreover, the natural homeomorphism $\mathbb{R}^n \oplus \mathbb{R}^m \cong \mathbb{R}^{n+m}$ lifts to a homeomorphism $S^n \wedge S^m \cong S^{n+m}$, for any n,m.

Definition 1.1. Given a based space X, we call $X \wedge S^1$ the *reduced suspension* of X. Sometimes we might write ΣX for $X \wedge S^1$. Also, we might write Ω^n for the functor $\widetilde{\mathrm{Hom}}_*(S^n,-)$. The isomorphism $S^n \wedge S^m \cong S^{n+m}$ translated into a natural isomorphism $\Omega^n \circ \Omega^m \cong \Omega^{n+m}$.

Another construction that will be useful is that of mapping cone.

Definition 1.2. Given a based map $f : X \to Y$, we denote by C_f the *mapping cone*, which is the pushout $(I_1 \wedge A) \vee_A B$, where I_1 is the unit interval $[0,1]$ with 1 as basepoint. Explicitly, this is the pushout in $\mathcal{T}op_*$ of the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle j}\downarrow & & \\ I_1 \wedge A & & \end{array}$$

j being the map $a \mapsto 0 \wedge a$.

Remark 1.3. When talking about based CW complexes, the basepoint will always be a 0-cell. This ensures, for example, that given based CW complexes X, Y, the subspace $X \times \{y\} \cup \{x\} \times Y$ is a CW subcomplex of $X \times Y$ and so $X \wedge Y$ inherits a based CW structure.

1.2. **Groups.** Given any natural number n, denote by Σ_n the (discrete) group of permutations of $\{1, \cdots, n\}$. Given n, m, there is a group homomorphism $\Sigma_n \times \Sigma_m \hookrightarrow \Sigma_{n+m}$ induced by the identification of $\{1, \cdots, n+m\}$ with $\{1, \cdots, n\} \bigsqcup \{1, \cdots, m\}$. Given a pair $(\sigma, \tau) \in \Sigma_n \times \Sigma_m$, denote the resulting permutation in Σ_{n+m} by $\sigma + \tau$.

Remark 1.4. Σ_n acts naturally on $\mathbb{R}^n$ by permuting coordinates. This action is linear and orthogonal, hence gives a group homomorphism $\Sigma_n \hookrightarrow O(n)$. The same action on $\mathbb{C}^n$ gives a group homomorphism $\Sigma \hookrightarrow U(n)$. By Remark 1.2, the orthogonal action of Σ_n on $\mathbb{R}^n$ transfers to a based, left action on S^n.

Definition 1.3. Let **Ab** the category of abelian groups and $\otimes$ the tensor product of abelian groups as $\mathbb{Z}$-modules. Denote by $\mathbf{Ab}^{\mathbb{Z}}$ the category of $\mathbb{Z}$-graded abelian groups.

2. SPECTRA

Definition 2.1 ([9])**.** A *symmetric spectrum* consists of the following data:

(1) A pointed Σ_n-space X_n, for all $n \geq 0$.
(2) A map of pointed spaces $\sigma_n : X_n \wedge S^1 \to X_{n+1}$, for all n. We'll refer to the maps σ_n as *structure maps*.

such that, for all $n, m \geq 0$, the map

$$X_n \wedge S^m \xrightarrow{\sigma_n \wedge \mathrm{id}} X_{n+1} \wedge S^{m-1} \xrightarrow{\sigma_{n+1} \wedge \mathrm{id}} \cdots \xrightarrow{\sigma_{n+m-2} \wedge \mathrm{id}} X_{n+m-1} \wedge S^1 \xrightarrow{\sigma_{n+m-1}} X_{n+m}$$

is $\Sigma_n \times \Sigma_m$-equivariant, where $\Sigma_n \times \Sigma_m$ acts on the target X_{n+m} by restriction of the action by Σ_{n+m}. A morphism of symmetric spectra $f : X \to Y$ consists of a Σ_n-equivariant map $f_n : X_n \to Y_n$, for every n, that are compatible with the structure maps, i.e. such that the following diagram commutes:

$$\begin{array}{ccc} X_n \wedge S^1 & \longrightarrow & X_{n+1} \\ {\scriptstyle f_n \wedge S^1}\downarrow & & \downarrow{\scriptstyle f_{n+1}} \\ Y_n \wedge S^1 & \longrightarrow & Y_{n+1} \end{array}$$

Denote by $\mathcal{S}p$ the category of symmetric spectra.

Definition 2.2. Given a symmetric spectrum X and $n \in \mathbb{Z}$, the *n-th naive homotopy group of X* is defined as the colimit of the sequence

$$\cdots \xrightarrow{K_{l-1}} \pi_{n+l}(X_l) \xrightarrow{K_l} \pi_{n+l+1}(X_{l+1}) \xrightarrow{K_{l+1}} \cdots$$

where K_l is defined as the composition of suspension $\pi_{n+l}(X_l) \to \pi_{n+l+1}(X_l \wedge S^1)$ followed by the induced map $(\sigma_l)_* : \pi_{n+l+1}(X_l \wedge S^1) \to \pi_{n+l+1}(X_{l+1})$. The colimit is denoted $\hat{\pi}_n(X)$.

Remark 2.1. Recall that $- \wedge S^1$ has a right adjoint given by Ω (the based loop space functor). In particular, the structure maps $X_n \wedge S^1 \to X_{n+1}$ are equivalent to maps $X_n \to \Omega X_{n+1}$. A symmetric spectrum is a Ω-*spectrum* if $X_n \to \Omega X_{n+1}$ is a weak homotopy equivalence, for all n.

Example 2.1. Given a spectrum X and a pointed topological space K, denote by $K \wedge X$ the spectrum with $(K \wedge X)_n := K \wedge X_n$ and structure map $K \wedge X_n \wedge S^1 \xrightarrow{K \wedge \sigma_n} K \wedge X_{n+1}$. Also, denote by X^K the spectrum with $(X^K)_n := \widetilde{\mathrm{Hom}}_*(K, X_n)$ and structure map

$$\widetilde{\mathrm{Hom}}_*(K, X_n) \wedge S^1 \to \widetilde{\mathrm{Hom}}_*(K, X_n \wedge S^1) \xrightarrow{(\sigma_n)_*} \widetilde{\mathrm{Hom}}_*(K, X_{n+1})$$

the first map sending (f, x) to $(f \wedge x)(t) = f(t) \wedge x \in X_n \wedge S^1$. For any space K, the functors $(K \wedge -, -^K) : \mathcal{S}p \to \mathcal{S}p$ are adjoint. We denote $-^{S^1} : \mathcal{S}p \to \mathcal{S}p$ by Ω, as in topological spaces.

Example 2.2. Given a morphism $f : X \to Y$ of spectra, denote by C_f the *mapping cone spectrum* with $(C_f)_n := C_{f_n} = (I_1 \wedge X_n) \vee_{X_n} Y_n$. The structure maps are defined using that $- \wedge S^1$ commutes with colimits, being a left adjoint, hence

$$C_{f_n} \wedge S^1 \cong (I_1 \wedge X_n \wedge S^1) \vee_{X_n \wedge S_1} (Y_n \wedge S_1) \to (I_1 \wedge X_{n+1}) \vee_{X_{n+1}} Y_{n+1}$$

the last map induced by the structure maps of the spectra X, Y.

Example 2.3. Denote by $\mathbb{S}$ the *sphere spectrum* with $\mathbb{S}_n := S^n$ and structure map the afromentioned based homeomorphism $S^n \wedge S^1 \cong S^{n+1}$. Given a topological space X, denote by $X \wedge \mathbb{S}$, called the *suspension spectrum* of X. In particular, we write $\mathbb{S}^n := S^n \wedge \mathbb{S}$. The induced functor $\Sigma^\infty : \mathcal{T}op_* \to \mathcal{S}p$ is left adjoint to $\Omega^\infty : X \mapsto X_0$, in particular $\mathrm{Hom}_{\mathcal{S}p}(\mathbb{S}, X)$ is isomorphic to the set of points of X_0.

Example 2.4. Given a based topological space X and abelian group A, denote by $A[X]$ the free abelian group generated by finite, A-linear combinations of the points of X, modulo the A-linear subgroup generated by the basepoint. The topology of this space is determined by the sequence of maps

$$A^n \times X^n \to A[X], \qquad (a_1, \cdots, a_n, x_1, \cdots, x_n) \mapsto \sum_k a_k \cdot x_k$$

For all n, the space $A[S^n]$ is an Eilenberg-Mac Lane space of type (A, n), see [2]. Finally, denote by HA the *Eilenberg-Mac Lane spectrum* with $(HA)_n := A[S^n]$ and structure maps $A[S^n] \wedge S^1 \to A[S^n \wedge S^1] \cong A[S^{n+1}]$, the first being

$$(\sum_x a_x \cdot x, y) \mapsto \sum_x a_x \cdot (x \wedge y)$$

Using that $A[S^n]$ is an Eilenberg-Mac Lane space of type (A, n), we can deduce that $\hat{\pi}_n(HA) \cong A$, if $n = 0$, and vanishes otherwise.

Example 2.5. Given a compact topological group G, denote by EG the total space of the universal principal G-bundle, which can be constructed as the geometric realization of the simplicial *space* $[n] \mapsto G^{n+1}$, where face maps are induced by projections. Consider $G = O(n)$ (as $O(0)$ we take the trivial group), the group of orthogonal automorphisms of $\mathbb{R}^n$, which acts on S^n via the one-point compactification of the left action on $\mathbb{R}^n$. Denote by MO the *real Thom spectrum* with

$$MO_n := EO(n)_+ \wedge_{O(n)} S^n$$

which is the quotient of $EO(n)_+ \wedge S^n$ by the right $O(n)$ action induced by the right action on $EO(n)_+$ and the left action on S^n. The group $O(n)$ then acts on MO_n by acting on $EO(n)_+$ on the left and so does Σ_n via the inclusion $\Sigma_n \to O(n)$ induced by the Σ_n action on $\mathbb{R}^n$ by permutation of the coordinates.

Letting $O(n)$ act on the first n components of $\mathbb{R}^{n+1}$ induces a group homomorphism $O(n) \to O(n+1)$, which then induces a continuous map $EO(n) \to EO(n+1)$. The structure map of MO is given by

$$(EO(n)_+ \wedge_{O(n)} S^n) \wedge S^1 \cong EO(n)_+ \wedge_{O(n)} S^{n+1} \to EO(n+1)_+ \wedge_{O(n+1)} S^{n+1}$$

where S^{n+1} in the middle term is a $O(n)$-space via the afromentioned inclusion $O(n) \to O(n+1)$. The isomorphism is given by $S^n \wedge S^1 \cong S^{n+1}$, which is $O(n)$-equivariant.

To construct the complex Thom spectrum MU, we need to work a bit harder. The issue stems from the fact that complex spheres correspond to even dimensional real spheres.

Example 2.6. Given a vector space V, denote by S^V its one-point compactification (in particular, $S^{\mathbb{R}^n} = S^n$). Denote by $\widetilde{MU}$ sequence of spaces $\widetilde{MU}_n := EU(n)_+ \wedge_{U(n)} S^{\mathbb{C}^n}$. As in the real case, the Σ_n action is induced by permutation of the factors in $\mathbb{C}^n$ and acts on $\widetilde{MU}_n$ by left action on $EU(n)_+$. Denote by MU the *complex Thom spectrum* with

$$MU_n = \widetilde{\mathrm{Hom}}_*(S^n, \widetilde{MU}_n) = \Omega^n(\widetilde{MU}_n)$$

with the action of Σ_n by conjugation. Consider then the map $\tilde{\sigma}_n : \widetilde{MU}_n \wedge S^{\mathbb{C}} \to \widetilde{MU}_{n+1}$ defined as the structure maps for the real Thom spectrum (like for $O(n)$, there is an embedding $U(n) \to U(n+1)$ and corresponding map $EU(n) \to EU(n+1)$). The structure maps of MU are then defined as via the map

$$MU_n \wedge S^2 = \widetilde{\mathrm{Hom}}_*(S^n, \widetilde{MU}_n) \wedge S^{\mathbb{C}} \to \widetilde{\mathrm{Hom}}_*(S^n, \widetilde{MU}_n \wedge S^{\mathbb{C}}) \xrightarrow{(\tilde{\sigma}_n)_*} \widetilde{\mathrm{Hom}}_*(S^n, \widetilde{MU}_{n+1})$$

Using $S^2 \cong S^1 \wedge S^1$ and the adjunction $(S^1 \wedge -, \Omega)$, together with the natural isomorphism $\Omega \circ \Omega^n \cong \Omega^{n+1}$, we get the desired structure map $MU_n \wedge S^1 \to \Omega^{n+1}(\widetilde{MU}_{n+1}) = MU_{n+1}$.

2.1. **Smash product.** Contrary to a simpler model for spectra (that is, sequential spectra), symmetric spectra have a symmetric monoidal structure given by *smash product*, of which $\mathbb{S}$, the sphere spectrum, is the monoidal unit. The explicit construction of $\wedge$ can be found in [9]. Here we simply recall the data of a commutative monoid in this structure, also called a *ring spectrum*.

Definition 2.3. A *ring spectrum* structure on a spectrum X consists of the following data:

(1) A unit element $u \in X_0$ (equivalent to a pointed map $S^0 \to X_0$ or spectra morphism $\mathbb{S} \to X$).
(2) For all n, m, a $\Sigma_n \times \Sigma_m$-equivariant *multiplication map* $\mu_{n,m} : X_n \wedge X_m \to X_{n+m}$.

subject to a certain set of conditions, such as associativity, unitarity, etc.

Example 2.7. Consider the sphere spectrum $\mathbb{S}$, take as unit $S^0 \to \mathbb{S}_0 = S^0$ the identity. As multiplication map take the homeomorphism $S^n \wedge S^m \cong S^{n+m}$. The sphere spectrum acts as $\mathbb{Z}$ for abelian groups, in that, for every ring spectrum X, there is only one right spectrum homomorphism $\mathbb{S} \to X$, namely the one induced by the unit element u.

Example 2.8. Given an abelian group A underlying a ring, recall the definition of the Eilenberg-Mac Lane spectrum HA. The unit $1 \in A$ induces a map $S^0 \to A[S^0]$ by $x \mapsto 1 \cdot x$. Multiplication are defined as $A[S^n] \wedge A[S^m] \to A[S^n \wedge S^m] \cong A[S^{n+m}]$, the first map being

$$(\sum_x a_x \cdot x, \sum_y a_y \cdot y) \mapsto \sum_{x,y} (a_x a_y) \cdot (x \wedge y)$$

Example 2.9. Recall the real Thom spectrum MO. The unit is given by the identity $S^0 \to MO_0 = S^0$. The multiplication map $\mu_{n,m} : MO_n \wedge MO_m \to MO_{n+m}$ is induced by isomorphism $\mathbb{R}^n \oplus \mathbb{R}^m \cong \mathbb{R}^{n+m}$ together with the morphism

$$EO(n)_+ \wedge EO(m)_+ \cong E(O(n)_+ \wedge O(m)_+) \to EO(n+m)_+$$

the last map induced by the group homomorphism $O(n) \times O(m) \to O(n+m)$, while the homeomorphism comes from geometric realization preserving products.

The same construction induces multiplication maps $\tilde{\mu}_{n,m} : \widetilde{MU}_n \wedge \widetilde{MU}_m \to \widetilde{MU}_{n+m}$, which induces a multiplicative struture on MU as the composition

$$\widetilde{\mathrm{Hom}}_*(S^n, \widetilde{MU}_n) \wedge \widetilde{\mathrm{Hom}}_*(S^m, \widetilde{MU}_m) \xrightarrow{\wedge} \widetilde{\mathrm{Hom}}_*(S^{n+m}, \widetilde{MU}_n \wedge \widetilde{MU}_m) \xrightarrow{(\tilde{\mu}_{n,m})_*} \widetilde{\mathrm{Hom}}_*(S^{n+m}, \widetilde{MU}_{n+m})$$

The unit of MU is also simply the identity $S^0 \to MU_0 = \mathrm{Hom}_*(S^0, S^0) \cong S^0$.

2.2. **Stable homotopy groups, stable weak equivalences and cohomology theories.** Symmetric spectra are one of the many models for the *stable homotopy category* $\mathcal{SHC}$. Originally constructed by Boardman ([3, 1]). This category comes with a symmetric monoidal structure, but proving the formal properties of this product is a lengthy process ([1]). Moreover, topologists attempted to construct a category $\mathcal{X}$ that would return the stable homotopy category upon localization at some class of weak equivalences, but that was symmetric monoidal before localization, symmetric spectra satisfying the description.

The downside of symmetric spectra is that introducing stable weak equivalences is not as straight-forward as in other models (such as *sequential* or *orthogonal* spectra). For example, if we take $\hat{\pi}$-equivalences (morphisms inducing isomorphisms of naive homotopy groups) and localize $\mathcal{S}p$ at them, we don't obtain $\mathcal{SHC}$. However, we can introduce stable weak equivalences by first looking at *simplicial* symmetric spectra (symmetric spectra with pointed simplicial sets instead of pointed spaces), then use the based singular simplicial set $Sing : \mathcal{T}op_* \to s\mathcal{S}et_*$ to transfer back stable weak equivalences to $\mathcal{S}p$ (see [4, §6.2]). The silver lining is that there is a full subcategory of spectra where stable equivalences are the same as $\hat{\pi}$-equivalences, these are *semistable spectra* ([9]) and every example given here is a semistable spectrum. In particular, the naive homotopy groups of semistable spectra are the *genuine* homotopy groups.

Remark 2.2. The smash product $\wedge$ and suspension $S^1 \wedge -$ on $\mathcal{S}p$ descends to a derived smash product $\wedge^L$ and derived suspension $\mathbb{S}^1 \wedge^L -$ on $\mathcal{SHC}$. Moreover, $\mathbb{S}^1 \wedge -$ is an equivalence of categories, hence there is, for every $n \geq 0$, a spectrum solving $\mathbb{S}^n \wedge^L X \cong \mathbb{S}$, we call this spectrum (unique up to isomorphism) $\mathbb{S}^{-n}$. Moreover, $\mathcal{SHC}$ has binary products and coproducts. Finally, $\mathcal{SHC}$ is additive (enriched over abelian groups).

Now, the stable homotopy category, similar to the derived category of an abelian category, underlies the structure of a triangulated category. The definition of a triangulated category can be found in [5], but it basically consists of an additive category $\mathcal{T}$ with a equivalence $\Sigma : \mathcal{T} \to \mathcal{T}$ together with a class of *distinguished triangles*. A triangle consists of a diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

A morphism of triangles is a commutative diagram as follows

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ \downarrow{\scriptstyle \alpha} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \Sigma\alpha} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X' \end{array}$$

Definition 2.4. An *elementary distinguished triangle* in $\mathcal{SHC}$ is the image under λ of a cofiber sequence $X \xrightarrow{f} Y \to C_f \to S^1 \wedge X$, for any morphism f (recall Example 2.2). A distinguished triangle in $\mathcal{SHC}$ is any triangle isomorphic to an elementary distinguished triangle.

Definition 2.5. An additive functor $E : \mathcal{SHC}^{op} \to \mathbf{Ab}$ is *cohomological* if it preserves products and, for every distinguished triangle $X \to Y \to Z \to \mathbb{S}^1 \wedge^L X$, the induced sequence $E(\mathbb{S}^1 \wedge^L X) \to E(Z) \to E(Y) \to E(X)$ is exact. Given $n \in \mathbb{Z}$ and a spectrum X, we write $E^n(X) := E(\mathbb{S}^{-n} \wedge^L X)$. Given a pointed topological space K, define $E(K) := E(\mathbb{S} \wedge K)$

Theorem 2.1. *Every cohomological functor is isomorphic to one of the form* $\mathrm{Hom}_{\mathcal{SHC}}(-, E)$*, for some spectrum* E*.*

In particular, if E underlies a ring spectrum, the presheaf $\mathrm{Hom}_{\mathcal{SHC}}(-, E)$ inherits a multiplicative structure: Up to stable weak equivalence, the multiplicative structure of E in $\mathcal{S}p$ transfers to a multiplicative structure of E in $\mathcal{SHC}$ ([9, Theorem 3.1]), meaning $\lambda : \mathcal{S}p \to \mathcal{SHC}$ is lax monoidal. Given spectra X, Y, the multiplication $\mu : E \wedge^L E \to E$ in $\mathcal{SHC}$ induces

$$E(X) \otimes E(Y) \to \mathrm{Hom}_{\mathcal{SHC}}(X \wedge^L Y, E \wedge^L E) \xrightarrow{\mu_*} \mathrm{Hom}_{\mathcal{SHC}}(X \wedge^L Y, E) = E(X \wedge^L Y)$$

A cohomological functor equipped with a pairing $E(X) \otimes E(Y) \to E(X \wedge^L Y)$ is called *multiplicative*. In particular, since $\mathbb{S} \wedge^L \mathbb{S} \cong \mathbb{S}$, the group $E(\mathbb{S})$ is actually a ring, with the unit $\mathbb{S} \to E$ of the ring spectrum translating into the unit of $E(\mathbb{S})$.

Lemma 2.1. *$E(\mathbb{S}) \cong \pi_0(E)$. More generally, $E^n(\mathbb{S}) \cong \pi_{-n}(E)$.*

Definition 2.6. Given a cohomological functor $E : \mathcal{SHC}^{op} \to \mathbf{Ab}$ and a pointed space X, denote by $\tilde{E}(X) := E(X \wedge \mathbb{S})$. In particular, $\tilde{E}(S^0) = E(\mathbb{S}) \cong \pi_0(E)$.

3. Complex Oriented Cohomology Theories

Given a vector bundle $F \to X$, we identify X with its image under the zero section. Given a point $x \in X$, denote by F_x its fiber.

Definition 3.1. Given a vector bundle $F \to X$, a *Thom class* consists of a cohomology class $t \in H^n(F, F - X, \mathbb{Z})$ such that, the restriction $t|_{F_x} \in H^n(F_x, F_x - \{0\}, \mathbb{Z}) \cong H^n(\mathbb{R}^n, \mathbb{R}^n - \{0\}, \mathbb{Z}) \cong \mathbb{Z}$ is one of the generators of $\mathbb{Z}$.

Remark 3.1. Thom classes are connected to the notion of Thom space of a vector bundle: Let $F \to X$ be a vector bundle, then $MF := F \cup \{\infty\}$ is the one-point compactification of the total space of the bundle. The pair (MF, ∞) is homotopy equivalent to the pair $(F, F - X)$, hence the existence of a Thom class is equivalent to the existence of a class in $t \in H^n(MF, \infty, \mathbb{Z}) = \tilde{H}^n(MF, \mathbb{Z})$ such that, for every $x \in X$, the pullback of t along the inclusion $S^n \cong F_x \cup \{\infty\} \subseteq MF$ induces a generator of $\tilde{H}^n(S^1, \mathbb{Z}) \cong \mathbb{Z}$.

The existence of a Thom class is equivalent to orientability of the bundle E, in the sense that $GL(F) \to X$ (the frame bundle of F) has two connected components, see [11]. Now, not all real bundles have a Thom class (since not all are orientable), but all complex vector bundles have orientable underlying real bundles. The point of a complex oriented cohomology theory is that every complex bundle $F \to X$ has a *natural* choice of Thom class t_F (see [7, Definition 2.22]). By the splitting principle ([11]), existence of Thom classes for all complex vector bundles can be reduced to complex line bundles first, then reduced to existence of a Thom class for the universal line bundle $L = EU(1) \times_{U(1)} \mathbb{C} \to \mathbb{C}P^\infty = BU(1)$ (where $EU(1)$ is the universal principal $U(1)$-bundles).

Theorem 3.1 ([1, Example 2.1])**.** *The zero section $z : \mathbb{C}P^\infty \to ML$ is a weak homotopy equivalence. In particular, the choice of a Thom class $\in \tilde{H}^2(ML, \mathbb{Z})$ is equivalent to the choice of a class $\in \tilde{H}^2(\mathbb{C}P^\infty, \mathbb{Z})$ such that, its pullback along the sphere $S^2 = \mathbb{C}P^1 \to \mathbb{C}P^\infty$ is a generator for $\tilde{H}^2(S^2, \mathbb{Z}) \simeq \mathbb{Z}$.*

Proof. First of all, $\mathbb{C}P^\infty$ is connected and all fibers of L are homeomorphic (and so their one-point compactifications are based homeomorphic), so the condition on a Thom class (generates the cohomology of the sphere $S^2 \cong ML_x \hookrightarrow ML$ under pullback) can be verified by looking at *one* point. In particular, we take that point to be $[\mathbb{C}] \in \mathbb{C}P^\infty$ (the complex line itself, viewed as a point of the infinite complex projective space).

The point $[\mathbb{C}]$ then induces a map $j : \mathbb{C}P^1 \to \mathbb{C}P^\infty$ and a fiber map $i : \mathbb{C} \to L_{[\mathbb{C}]} \subseteq L$, and so $Mi : \mathbb{C}P^1 \to ML$. Under the weak homotopy equivalence $\mathbb{C}P^\infty \to ML$, the map j and Mi coincide (see [6, Theorem 3.9]). In particular, $t \in \tilde{H}^2(ML, \mathbb{Z})$ is a Thom class, if and only if, the corresponding class $c = z^* t \in \tilde{H}^2(\mathbb{C}P^\infty, \mathbb{Z})$ satisfies that $j^* c \in \tilde{H}^2(\mathbb{C}P^1, \mathbb{Z}) \cong \mathbb{Z}$ is a generator. □

The class $c \in \tilde{H}^2(\mathbb{C}P^1, \mathbb{Z})$ is nothing more than the first Chern class of the canonical line bundle. In general, a complex orientable cohomology theory is a generalized cohomology theory with the choice of a generalized first Chern class.

Definition 3.2. Given a ring spectrum E (we denote by E also the corresponding cohomology theory), a *complex orientation* consists of a class $c_1^E \in \tilde{E}^2(\mathbb{C}P^\infty)$ such that, its pullback along $S^2 \cong \mathbb{C}P^1 \to \mathbb{C}P^\infty$ is a generator for $\tilde{E}^2(S^2) \cong \tilde{E}(S^0) \cong \pi_0(E)$. If a complex orientation exists, we call E *complex orientable*.

Remark 3.2. The isomorphism $\tilde{E}^2(S^2) \cong \tilde{E}(S^0)$ comes from the suspension isomorphism $E^k(\mathbb{S}^n \wedge X) \cong E(\mathbb{S}^{-k} \wedge \mathbb{S}^n \wedge X) \cong E(\mathbb{S}^{-k+n} \wedge X)$ together with $S^2 \wedge \mathbb{S} \cong \mathbb{S}^2$ (plus, recall that E^0 is E, by definition, since $\mathbb{S}^0 = \mathbb{S}$).

The existence of a complex orientation has a number of consequences. First, any complex vector bundle $F \to X$ have a E-Thom class, i.e. a class $t \in \tilde{E}^n(MF)$ such that pullback along any fiber $i_x : \mathbb{S}^n \cong MF_x \hookrightarrow MF$ is a generator of $\tilde{E}^n(S^n) \cong \pi_0(E)$. Next, we can fully calculate the cohomology of $\mathbb{C}P^\infty$ as follows: Recall that E being a ring spectrum implies that $\pi_0(E)$ is a ring and the 0-component of the graded ring $\pi(E) = \bigoplus_{n\in\mathbb{Z}} \pi_n(E)$. Let $\mathbb{C}P^\infty_+ = \mathbb{C}P^\infty \bigsqcup \{+\}$, where $+$ is the new basepoint, then we can pullback c_1^E along the natural map $\mathbb{C}P^\infty_+ \to \mathbb{C}P^\infty$ (sending $+$ to the original basepoint of $\mathbb{C}P^\infty$) to get a class that we also write as $c_1^E \in \tilde{E}^2(\mathbb{C}P^\infty_+)$.

Lemma 3.1. *The class $c_1^E \in \tilde{E}^2(\mathbb{C}P^\infty_+)$ induces an isomorphism $\pi(E)[\![x]\!] \to \tilde{E}^*(\mathbb{C}P^\infty_+)$ of $\pi(E)$-modules.*

For brevity, we'll write $E^*(\mathbb{C}P^\infty)$ to mean $\tilde{E}^*(\mathbb{C}P^\infty_+)$. Finally, there is a importart example of complex oriented ring spectrum: The complex Thom spectrum. Notice that ML, the Thom spectra of $EU(1) \times_{U(1)} \mathbb{C}$ is exactly $\widetilde{MU}_1$, in particular the zero section is a homotopy equivalence $z : \mathbb{C}P^\infty \to \widetilde{MU}_1$. This map can be used to construct a morphism of spectra representing a class in $c_1^{MU} \in \mathrm{Hom}_{\mathcal{SHC}}(\mathbb{S}^{-2}\mathbb{C}P^\infty, MU)$ satisfying the condition for complex orientations.

Theorem 3.2. *Given a ring spectrum E, a ring spectra morphism $f : MU \to E$ induces a complex orientation $c_1^E = f_*(c_1^{MU})$, where f_* is the post-composition map $\mathrm{Hom}_{\mathcal{SHC}}(-, MU) \to \mathrm{Hom}_{\mathcal{SHC}}(-, E)$. This map*

$$\mathrm{Hom}(MU, E) \to \{\text{complex orientations on } E\}$$

is a natural bijection.

In this sense, (MU, c_1^{MU}) is the universal complex oriented ring spectrum.

4. Formal Group Law

Now, we look at the final piece. Formal group laws. In the case of $\mathbb{Z}$-valued singular cohomology, the first Chern class is the usual one. In particular, it satisfies the additional property that

$$c_1(L \otimes K) = c_1(L) + c_1(K)$$

for any two line bundles L, K (over some space X). This is not a property that a generic Chern class c_1^E might satisfy (see [8, Example 7]), but $c_1^E(L \otimes K)$ will be some $\pi(E)$-linear combination of powers of $c_1^E(L)$ and $c_1^E(K)$. To see this, consider the map $m : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ given by tensor product of lines: A point in $\mathbb{C}P^\infty$ is represented by some non-zero point $(z_0, \cdots, z_n) \in \mathbb{C}^{n+1}$, for some n, then m is defined by taking representatives $(z_0, \cdots, z_n)$ and $(w_0, \cdots, w_m)$ and return the point in $\mathbb{C}P^\infty$ represented by

$$(z_i w_j)_{i,j} \in \mathbb{C}^{(n+1)(m+1)}$$

If we identify line bundles with homotopy classes of maps into $\mathbb{C}P^\infty$, the monoid structure induced on homotopy classes by m is equivalent to the monoidal structure given by tensor product of line bundles.

Take now a complex oriented cohomology theory, we saw that $E^*(\mathbb{C}P^\infty) \cong \pi(E)[\![x]\!]$, with the choosen orientation c_1^E being a generator. Then $E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong \pi(E)[\![x, y]\!]$, where generators now are the pullback of c_1^E along the two projections $p_1, p_2 : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$. If we pullback c_1^E along m, we obtain an expression

$$F(p_1^* c_1^E, p_2^* c_1^E) = m^*(c_1^E)$$

Definition 4.1. Given a ring R, a *formal group law* is a formal power series $F \in R[\![x,y]\!]$ such that:

$$F(x,0) = x, \qquad F(x,F(y,z)) = F(F(x,y),z)$$

and $F(x,y)$ is invariant under the automorphism of $R[\![x,y]\!]$ exchanging x,y. If R is graded, a *graded* formal group law is one where the coefficient of $a_{n,k} \in R$ has degree $2k+2n-2$, for all n,k.

Theorem 4.1. *The formal power serie $F(x,y)$ is a graded formal group law for $R = \pi(E)$.*

The reason for $m^*(c_1^E)$ to be a graded formal group law boils down to $\mathbb{C}P^\infty$ being a homotopy commutative, topological monoid, where the basepoint acts as the unit of m, see [7]. Therefore, a complex orientation induces a graded FGL on $\pi(E)$. In particular, the complex orientation on MU induces a formal group law for $\pi(MU)$. This way, given a ring map $MU \to E$, we can first pushforward the complex orientation on MU to one on E, then consider the associated graded FGL for $\pi(E)$, or we can take the induced map $\pi(MU) \to \pi(E)$ and the formal group law $F^{MU} \in \pi(MU)[\![x,y]\!]$ and base change to a formal group law on $\pi(E) \otimes (\pi(MU)[\![x,y]\!]) \cong \pi(E)[\![x,y]\!]$. This two processes are one and the same.

Theorem 4.2 (Quillen)**.** *Given a (graded) ring R, morphisms of (graded) rings $\pi(MU) \to R$ are in bijection with (graded) formal group laws for R. The formal group law F^{MU} induced by c_1^{MU} is the universal (graded) FGL.*

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